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Dispensabilism

I. Field's Project

Benacerraf presented a dilemma for all philosophical accounts of mathematics: either they conflict with our best epistemology or they conflict with our best theories of truth and semantics.
Quine's indispensability argument avoids Benacerraf's dilemma by formulating a new epistemology for mathematics, one that is consistent with standard semantics for mathematical sentences.

QI
QI1: We should believe the theory which best accounts for our sense experience.
QI2: If we believe a theory, we must believe in its ontological commitments.
QI3: The ontological commitments of any theory are the objects over which that theory first-order quantifies.
QI4: The theory which best accounts for our sense experience first-order quantifies over mathematical objects.
QIC: We should believe that mathematical objects exist.

At the end of our discussion of Quine's indispensability argument, I raised some worries about Quine's holism, and his method for determining the ontological commitments of a theory.
Those worries were about the first three premises of QI, but we are not going to pursue them, now.
Field's work is the most influential response to QI, and it is aimed at the fourth premise.
Field claims that the theory which best accounts for our experience need not quantify over mathematical objects.
If Field is right that QI4 is false, Quine's indispensability argument fails, and Benacerraf's dilemma re-emerges.

At the heart of Field's project is his proposed nominalistic reformulation of Newtonian Gravitational Theory (NGT).
Field tries to show that we can replace the quantification over mathematical objects which appears in first-order versions of NGT with quantification over space-time points or regions.
We need space-time points and regions in order to formulate scientific theories.
If we can use them to do the work that the mathematical objects normally do, then we can eliminate mathematical axioms from our best theory, and deny QI4.

Specifically, Field seeks to replace the bridge functions used to represent measurements of quantities like mass and velocity with axioms governing space-time.
He produces representation theorems that show how the space-time points and regions can do the work that mathematical objects do in standard theories.
Standard scientific theories, when axiomatized and formalized, will have three components.

1. A logical system, used for inference
2. Mathematical axioms
3. Scientific axioms

The mathematical axioms are included to provide mathematical language and inferences.
But, mathematical axioms can not be merely placed alongside the physical ones.

We need, in addition to the two sets of axioms, bridge functions which relate the theorems of mathematics to the theorems of science.
These bridge functions are the kinds of statements we ordinarily find in physical science when we measure a physical quantity, or when we state a physical law as a function.

> The speed of light $c = 3 \times 10^8$ cm/s^2
> $G = 8 \pi T$, where G is the gravitational tensor and T is the stress-energy tensor

Such laws include both mathematical and physical terms.
If we can eliminate the mathematical axioms and replace the mathematical terms in the theory with terms which refer to space-time points and regions, Field believes that we can stop believing that mathematical objects exist.

Field says that his construction is not directly an argument for nominalism.
Rather, it is a counter-argument to the indispensability argument.

> Nothing in this monograph purports to be a positive argument for nominalism. My goal rather is to try to counter the most compelling arguments that have been offered against the nominalist position (Field 4).

Still, Field's broader goal is to establish a version of nominalism which he calls fictionalism.
Fictionalism is the view that mathematical terms are empty names and mathematical sentences are either false, for existence claims, or vacuously true, for purely mathematical entailments.
'There are perfect numbers' is false.
'All rhombuses are parallelograms' is true, but only because there are no rhombuses.

Field aims at the indispensability argument, because he sees it as the most serious challenge to nominalism, and he accepts its major premises.

> The only non-question-begging arguments that I have ever heard for the view that mathematics is a body of truths all rest ultimately on the applicability of mathematics to the physical world (Field viii).

To avoid mathematical objects, some nominalists (other than Field) try to re-interpret mathematical axioms as referring to non-mathematical objects, like possible inscriptions.
Such strategies give up standard semantics for mathematical propositions.
The strength of Field's work is that it eliminates, rather than re-interprets, the mathematical axioms.
Field's theory interprets the mathematical axioms standardly, and then claims that mathematical claims are false.
Here is an argument for Field's fictionalism.

> FF1. We should take mathematical sentences at face value.
> FF2. If we take (some of them) to be non-vacuously true, then we have to explain our access to mathematical objects.
> FF3. The only good account of access is the indispensability argument.
> FF4. But, the indispensability argument fails.
> FFC. So, we should take non-vacuous mathematical sentences to be false.

FF1 adopts Benacerraf's claim, against those (e.g. Locke, Hilbert, and all of the other philosophers Benacerraf calls combinatorialists) who re-interpret mathematical sentences to be about something other than mathematical objects.
FF2 and FF3 seem to be mis-understanding Quine's epistemology.
On Quine's anti-reductionist method, there is no access problem.
Nevertheless, the importance of Field's program consists in FF4, in his argument against QI4..

Field's project consists of two parts.
First, he develops a nominalist counterpart to a standard scientific theory
Second, he tries to show that mathematics applies conservatively to nominalist theories, to assure us that nominalist counterparts are adequate substitutes.
Before we discuss these two parts of Field's project, we should look more closely at its framework.

II. Three Ground Rules for Nominalistic Reformulations of Standard Science

We need to establish some ground rules for reformulations such that the resulting theories can legitimately be considered nominalistic.
Three ground rules are fairly uncontroversial.

> GR.1: Adequacy: A reformulation must not omit empirical results of the standard theory.

For an inadequate theory, consider all the theorems of standard science which make no use of numerical constants.
In other words, we could easily nominalize science by just eliminating all our references to mathematical objects.
But, we wouldn't be allowed any measurement, or any functions.
The resulting theory would be laughably inadequate.

> GR.2: Logical Neutrality: A reformulation must not reduce ontology merely by extending logic, or ideology.

The logical framework of a theory can be more or less controversial.
At the least-controversial end, a theory can be cast in first-order logic.
First-order logic is complete, in the sense that all logically true formulas are derivable.
All truths are provable.
Gödel proved the completeness of first-order logic in his doctoral dissertation a year before he proved the incompleteness theorems for mathematics.
Moreover, first-order logic makes no existence assertions of its own, though it can clearly display the existence assertions of a more substantial theory using its language.

At the more-controversial end, a theory can invoke the full theory of second-order logic.
In second-order logic, quantifiers bind predicate variables, and range over properties.
Properties are, at best, sets, and at worst, intensional objects.
We can construct the real numbers by using sets, and we can represent functions using sets.
If we use full second-order logic we can avoid introducing mathematical axioms by using the supposedly second-order logical axioms.
Some logics use some second-order machinery, but not all of it.

Extensions of first-order logic other than second-order quantification have been invoked by nominalists.
I mentioned that reformulations which merely reinterpret the mathematical axioms are inadequate bases for nominalism.
One such type of reformulation interprets mathematical statements as about possible inscriptions.
Another takes mathematics to be about possible arrangements of objects.
Such modal reinterpretations force an extension of logic to include controversial modal operators.

> ◊P: P is possible
> □P: P is necessary

Actually, we need only one modal operator, as each can be defined in terms of the other.

> ◊P ≡ ~□~P
> □P ≡ ~◊~P

If a reformulation of scientific theory is to eliminate commitments to abstract objects, our real commitments must be found in the reformulated theory.
Theories which include modal logic can not rely on Quine's arguments that we find our commitments by looking at our regimented theory.
Such arguments only apply to Quine's canonical language of first-order logic, not to any kind of regimentation.
First-order logic makes no ontological commitments itself, but modal logics refer to possible worlds.
Philosophers have become increasingly comfortable with possible worlds in the wake of developments in model theory for modal logic, especially Kripke models.
In a Kripke model, '◊$\mathscr{F}$' is true in a world just in case there is an accessible possible world in which $\mathscr{F}$ is true.
Any appeal to possible worlds is problematic for the dispensabilist, since such claims conflict with the desire for parsimony that underlies both the indispensability argument and its dispensabilist response.

Reformulations which reject the indispensabilist's method for determining commitment by adopting modal logic do not succeed as dispensabilist strategies.
The addition of modal logic mitigates the significance of the elimination of quantification over mathematical objects.

> Avoidance of modalities is as strong a reason for an abstract ontology as I can well imagine (Quine, "Reply to Charles Parsons" 397).

Trying to avoid mathematical commitments by strengthening one's logic does not remove our ontological commitments.
That would be like magic.

> [I]t can be seen that there is something dubious about the practice of just helping oneself to whatever logical apparatus one pleases for purposes of nominalistic reconstruction while ignoring any customary definitions that would make the apparatus nominalistically unpalatable: for by doing so, one can make the task of nominalistic reconstruction absolutely trivial – and so absolutely uninteresting. (Burgess and Rosen, *A Subject with No Object* 175)

Quine calls second-order logic set theory in sheep's clothing, and that's when we interpret second-order

quantification in its least controversial way.
Thus, most formalizations of scientific theory try to rely on first-order logic, or only minor extensions of that logic.
Any reformulation must accept the logical neutrality of first-order logic if it is to serve as a response to QI.

> GR.3: Conservativeness: The addition of mathematics to the reformulated theory should license no additional nominalist conclusions.

I discuss conservativeness in more detail below; see §VI.
For now, let's just get the definition down.
Casually, it just means that adding mathematics to science does not generate new science.
Field believes that conservativeness is an essential characteristic of mathematics.

> Standard mathematics *might* turn out not to be conservative...for it might conceivably turn out to be inconsistent, and if it is inconsistent it certainly isn't conservative. We would however regard a proof that standard mathematics was inconsistent as extremely surprising, and as showing that standard mathematics needed revision. Equally, it would be extremely surprising if it were to be discovered that standard mathematics implied that there are at least 10^6 non-mathematical objects in the universe, or that the Paris Commune was defeated; and were such a discovery to be made, all but the most unregenerate rationalist would take this as showing that standard mathematics needed revision. *Good* mathematics *is* conservative; a discovery that accepted mathematics isn't conservative would be a discovery that it isn't good (Field 13).

Field provides a formal definition of conservativeness.
Let A be any nominalistically statable assertion, N any body of such assertions, and S any mathematical theory.
Take 'Mx' to mean that x is a mathematical object.
Let A* and N* be restatements of A and N with a restriction of the quantifiers to non-mathematical objects.
This restriction yields an agnostic version of the nominalist theory; it does not rule out the existence of mathematical objects.
S is conservative over N* if A* is not a consequence of N*+S+'∃x~Mx' unless A is a consequence of N.

Conservativeness is a metalogical notion.
There are two distinct approaches to metalogic: proof theory and model theory.
In proof theory, we study axioms and rules of inference.
In model theory, or semantics, we study interpretations, satisfaction, and truth.
For a complete theory, like first-order logic, proof theory and model theory yield the same results.
For an incomplete theory, proof theory and model theory diverge: there will be semantic consequences of the theory that are not provable.
Since conservativeness is a metalogical notion, and since it is applicable to strong-enough mathematical theories, there is both a proof-theoretic version and a model-theoretic version.
The proof-theoretic version, which Field calls deductive conservativeness, says that mathematics does not allow new theorems to be derived from the nominalistic theory.
The model-theoretic version, which Field calls semantic conservativeness, says that no additional statements come out true in any model of the theory which includes mathematics.

Conservativeness is essential to the dispensabilist argument for nominalism for two reasons.
First, it serves as a check on the adequacy of the nominalist reformulation.
If mathematics does not apply conservatively to NGT*, then the standard theory will yield more consequences.
The nominalist theory will be shown to omit some theorems of the standard theory.

Second, conservativeness provides an account of the applicability of mathematics to science.
Without this account, we must remain suspicious of the practical utility of mathematics, and its ubiquity in actual science.
Indeed, the platonist is particularly saddled with this worry.
Why would abstract objects, ones that do not exist spatially or temporally, be so useful, so effective, in our empirical theories?
The possibility of providing a satisfying account of the applicability of mathematics makes Field's project especially alluring.
He does not merely eliminate mathematics from scientific theory.
He attempts to show that our ordinary uses of mathematics are consistent with nominalist principles.

III. Attractiveness: A Fourth Ground Rule?

Field claims that the dispensabilist reformulation must be an attractive theory, in addition to obeying the other ground rules.

> GR.4: Attractiveness: The dispensabilist must show, "[T]hat one can always reaxiomatize scientific theories so that there is no reference to or quantification over mathematical entities in the reaxiomatization (*and one can do this in such a way that the resulting axiomatization is fairly simple and attractive*)" (Field viii, emphasis added).

At first, GR4 looks subjective.
Theories may be attractive or grotesque in a variety of ways.
Theories which have few axioms tend to have longer derivations.
Theories which provide elegant proofs tend to make more assumptions.
Which kind of theory we prefer may depend on our interests.

Field has specific constraints for GR.4 in mind.
He uses it to rule out some specific awkward reformulations of standard science.
For example, one could take all of the nominalistically-acceptable results of standard science as axioms.
Call this easy nominalization N_e.
N_e will not reduce diverse experiences to a few, simple axioms.
Still, N_e yields all the desired consequences of standard science, satisfying all the other ground rules.

> If no attractiveness requirement is imposed, nominalization is trivial... Obviously, such ways of obtaining nominalistic theories are of no interest (Field 41).

In other words, it can't be that easy to eliminate our beliefs about mathematics.

In order to determine if GR.4 is a legitimate criterion, we should understand what Field means by 'attractive', and how it applies to a potential reformulation.

If a reformulation must produce a useful theory for the practice of science, N_e is certainly unacceptable even though it satisfies GR.1, GR.2 and GR.3.
But, the standard theory regimented in first-order logic is similarly unacceptable.
No working scientist has any use for a fully regimented physical theory.
GR.4 is too strong a requirement, since the original theory violates it, too.
The reformulation, like the original regimented standard theory, is written to reveal commitments, not to be useful.
For Field, the irrelevance of attractiveness, taken as requirement that a reformulation should be useful to practicing scientists, should be especially obvious.
Field attempts to establish conservativeness for mathematics.
The point of arguing for the conservativeness of mathematics is to establish that it is acceptable to continue using standard science.
Field's goal is not to replace the working scientist's methods and tools, but to legitimate their use.
So, there is no justification for taking GR.4 as a requirement that reformulations be useful for scientists.

We might take attractiveness to be a demand that a reformulation have the same explanatory strength as the standard theory.
We can see that Field wants his reformulation to be explanatory in the way he defends his reformulations on the basis of a principle of intrinsic explanation.

> The elimination of numbers [from science], unlike the elimination of electrons, helps us to further a plausible methodological principle: the principle that *underlying every good extrinsic explanation there is an intrinsic explanation*. If this principle is correct, then real numbers (unlike electrons) have *got* to be eliminable from physical explanations, and the only question is how precisely this is to be done (Field 44).

Field supposes that the purposes of physical theory is to explain physical phenomena.
Demanding that a reformulation has explanatory merit again places too great a burden on it.
The standard theory it is meant to replace, when written to reveal commitments, can not be explanatory either.
No theory regimented into first-order logic can be explanatory, since it can not be perspicuous.
Compare the following two inferences.

IM I have two mangoes.
Andrés has three different mangoes.
So, together we have five mangoes.

IN $(\exists x)(\exists y)(Mx \bullet My \bullet Bxm \bullet Bym \bullet x{\neq}y)$
$(\exists x)(\exists y)(\exists z)(Mx \bullet My \bullet Mz \bullet Bxa \bullet Bya \bullet Bza \bullet x{\neq}y \bullet x{\neq}z \bullet y{\neq}z)$
$(x)[(Mx \bullet Bxa) \supset {\sim}Bxm]$
$\therefore (\exists x)(\exists y)(\exists z)(\exists w)(\exists v)(Mx \bullet My \bullet Mz \bullet Mw \bullet Mv \bullet x{\neq}y \bullet x{\neq}z \bullet x{\neq}w \bullet x{\neq}v \bullet y{\neq}z \bullet$
$y{\neq}w \bullet y{\neq}v \bullet z{\neq}w \bullet z{\neq}v \bullet w{\neq}v)$

IN is the first-order-logical cognate of the inference in IM.
But, IM is clearly more explanatory.
The gap between statements which are explanatory and those which are formalized only increases with the sophistication of science.

When we regiment either standard science or the dispensabilist reformulation, we show that in principle a theory may be written in terms of the basic logical vocabulary and as few primitives as possible.
We can imagine such regimentations and their domains of quantification, and use these to clarify our commitments.
While the regimented theory does not explain physical phenomena, it may form part of an account of those phenomena.
Given a phenomenon and a covering law, we could regiment the law and apply it to some initial conditions which yield, through syntactic transformations, the phenomenon.
Some elements of an account, like this imagined inference, need not be perspicuous, or otherwise useful.
But the original un-regimented theory is the one which does all the explanatory work.
We can not demand more of the reformulation, that it be explanatory, that we do of the original, first-order logical version of the theory.

The question at hand is whether GR.4 can be taken as a demand for an explanatory theory.
The dispensabilist construction may reveal our commitments, but at the cost of explanation.
Like practical utility, explanatory merit can not serve to cash out Field's attractiveness requirement.

GR.4 can not be a requirement for either practical utility or explanatory merit of the reformulated theory.
Reformulations are merely in-principle accounts like the original Quinean theory and need not be useful or explanatory.
Still, reformulations like N_e do not provide a sentence by sentence translation of the original theory which leaves the basic structure intact.
They do not reduce the laws to a neat and tidy few axioms.
Since Field insists on the importance of attractiveness, we can at least take GR.4 to be a requirement that the reformulations show how to translate axioms of standard science directly into nominalist language.
For more than that, we may have to adopt Potter Stewart's claim about obscenity: I know it when I see it.

IV. Space-Time Points

Field's reformulation of NGT quantifies over space-time points, or regions, in place of real numbers.
This is a natural place for concern about his project.
Field claims that space-time points are nominalistically acceptable, though mathematical objects are not.
Hillary Putnam, in an early review of Field's work denied that space-time points are nominalistically acceptable.

> Hartry Field has claimed...that one can do physics without reference to abstract entities. But his construction requires that we accept *absolute space-time points* and arbitrary sets of space-time points as 'concrete'; most philosophers (including myself) would regard this as 'cheating' (Putnam 1981: 175).

Field defends uses of space-time points, in part, on the basis of independent considerations in science.
In field theories, we reify space-time in order to ascribe field properties to something.
Electromagnetic and gravitational fields are depicted as having strength at all regions of space around a source.
In general relativity, for example, distortions created by the gravitational pull of massive bodies may be seen as curving space-time itself.
Even theories about electrons, and the atom itself, may be formulated as field theories, rather than about

discrete objects with particular, circumscribed spatio-temporal locations.
The force of a field persists even in a vacuum.
Thus, in order to ascribe forces to something, to have an object to which the predicate for force applies, we posit space-time points.

Philosophers who believe in the existence of space-time points and regions are called substantivalists.
Those who oppose reifying space-time are called relationalists.
The relationalist believes that the position of an object in space-time can only be described in terms of its relations to other objects, rather than in terms of its position in an absolute space-time.
The contemporary debate between substantivalists and relationalists is the heir to the eighteenth-century debate between Newton and Leibniz over absolute and relational space.

Appeals to the uses of space-time points in field formulations of physical theories do not settle the question of whether such points are nominalistically acceptable.
The same theories that refer to fields refer to mathematical objects.
It is difficult to see how to eliminate the mathematical objects; that's the motivation for Field's project.
But, some field formulations of some physical theories have alternate axiomatizations which do not appeal to space-time points.
In fact, we may be able to avoid positing space-time points by reifying fields themselves, taking them as objects.
We could ascribe to fields the same structure that Field ascribes to space-time points, the mathematical structure relevant to physical theory.
This suggestion, as regards Field's project, stems from David Malament.

If the mere fact that there are uses of space-time points in field formulations of physical theories were sufficient to make them acceptable to the nominalist, then the uses of mathematics would be acceptable as well.
We need a better argument for space-time points.
Field believes that Benacerraf's epistemological worry, that we have no explanation of our reliable knowledge of mathematical objects, remains a worry for the platonist, but not for the space-time substantivalist.
Field bases his defense of space-time points on a further claim that they are empirical posits, as opposed to *a priori* ones.

> Perhaps it is a bit odd to use the phrase 'physical entity' to apply to space-time points. But however this may be, space-time points are not abstract entities in any normal sense. After all, from a typical platonist perspective, our knowledge of mathematical structures of abstract entities (e.g. the mathematical structure of real numbers) is *a priori*; but the structure of physical space is an empirical matter (Field 31).

In later work, Field even argues that we have sensory access to space-time points.

> For there are quite unproblematic physical relations, viz., spatial relations, between ourselves and space-time regions, and this gives us epistemological access to space-time regions. For instance, because of their spatial relations to us, certain space-time regions can fall within our field of vision (Field 1982a: 68).

Sensory access to empty regions is implausible.
While a space-time region can fall within our field of vision, we can not actually see an unoccupied region of space.
We can not even see one that is filled with nitrogen and oxygen.
If we could see empty space, Leibnizian arguments for relationalism could not get started.
Still, worries about access to space-time points are moot because we can explain the origins of our beliefs in them in other ways.
We learn about physical theory through textbooks and lectures and such.
We justify these beliefs based on how the posit of space-time points helps account for our best physical theory.
Quine's metaphysical methodology, of constructing a theory and then modeling it to reveal our ontological commitments, renders the traditional access problem inert.
Field, in attacking QI4, accepts the method underlying the indispensability argument.

Field's argument that space-time points are empirical objects is more plausible than his claim that we have access to them, since we may take space-time points and regions as concrete objects.
Given Field's implicit acceptance of QI1-3, he need not claim sensory access to an empirical object to have knowledge of it.
I don't think that Putnam is correct that Field is cheating.
But, this criticism is worth keeping in mind as we proceed to discuss Field's project.

V. Space-Time and the Representation Theorems

We will not spend much time on the details of Field's nominalistic construction.
The basic idea is to choose some arbitrary regions of space-time to serve as the bases for measurement in place of the real numbers.
Field shows that we can impose a structure, including measurement (greater and less) and order (betweenness), on space-time regions.
Then, he shows, by proving representation theorems, that any uses of the real numbers can be replaced by the structure he describes for space-time points.
Representation theorems are standard functions which establish a homomorphism between two sets, or structures.
In his axiomatization for Euclidean geometry, Hilbert constructed representation theorems to map geometric points onto the real numbers.
Field maps space-time points onto real numbers.
More specifically, since we need four coordinates to represent a point in space-time (three spacial coordinates and one temporal coordinate), the representation theorems are supposed to map space-time points onto R^4.
Field's representation theorems explain how mathematics can be useful, given a nominalist theory, by showing that statements which use mathematics are convenient shorthand for nominalistically acceptable sentences about space-time points.

The representation theorems are constructed within a background theory which includes mathematics, since we need mathematics in the theorems, and for the mapping functions.
The representation theorems are not, therefore, available to the nominalist.
Their role is to convince the platonist that the nominalist theory is acceptable.
The platonist can see that any sentence of physical theory deducible using mathematics can be derived

without mathematics.
Thus, if Field is correct, platonism is self-refuting.

There are important questions about whether Field's technical work can succeed.
It is pretty clear that the theorems he provides do not do the work he intended.
In part, the problem is with his logic.

Field originally presented two versions of his reformulation.
First, he outlined a first-order version, which quantified over space-time regions.
The first-order version does not allow Field to construct his representation theorems.
In the first-order language, using tools familiar to those who have worked through Gödel's incompleteness proofs, we can construct a sentence, Con_N, asserting the consistency of the first-order nominalist theory, N.
Con_N is unprovable within N, but it is provable using a theory which includes a stronger set theory, S.
If there were a representation theorem which mapped the regions of space into R^4, then Con_N would be derivable from S+N.
That is, once we add set theory to the nominalist theory, Con_N, which follows from S, should follow from S+N.
Either there are no first-order representation theorems, or S is not conservative over N.
Without representation theorems, Field has not shown that his reformulation is adequate; it violates GR1.
Without conservativeness, Field's project violates GR2.

Penelope Maddy calls the first-order version of Field's reformulation inadequate because of its limited mathematical framework.

> The first order fictionalist adopts an anemic view of the scientific theory of space, one that pays closer attention to narrow evidential relations between particular sentences than to broader explanatory goals. This last may be its greatest weakness (Maddy 1990b: 203).

Maddy's claim seems to be that Field's theory is unattractive, in the explanatory sense.
In the first-order version, Field adopts axiom schemata for space-time regions, rather than full, second-order versions of axioms.
Since he only is able to include instances of the schemata in his reformulation, the nominalist theory is not as explanatory as the standard one.

Alongside the first-order version of his reformulation, and in greater detail, Field sketched a version using a weak version of second-order logic, called mereology, or the logic of sums, and a finite cardinality quantifier.
Field introduces second-order quantification to treat continuity.
I'll present the mathematical version of continuity and Field's replacement so you can get the idea.
Since many physical functions, like mass, temperature, and velocity, are assumed to be continuous, real numbers are used to represent their measurements.
In standard physics, the continuity of the reals is assumed via a topological axiom of continuity.
Define a set to be a *neighborhood* of a point (within a topological space) if it has a subset in the topology which contains the point.

Then continuity is defined in terms of neighborhoods.

> A function f is continuous at a point c if for every neighborhood V of f(c) there exists a neighborhood U of c such that $f(x) \in V$ whenever $x \in U$.

Field defines continuity in terms of space-time points or regions, for particular predicates which are presumed continuous.
Consider a primitive predicate like Temp-bet, defined over the space-time points.
'x Temp-Bet yz' says that x displays a temperature between those which y and z display.
'$x \approx_{Temp} y$' is short for 'x Temp-Bet yy,' and means that x and y display the same temperature.
Similar constructions can be designed for other scalar quantities, like Mass-Bet or Veloc-Bet.
(See Field 55-64, especially p 62.)
The continuity of Temp-bet is defined parallel to the version for real numbers.

> A region R is temperature-basic iff there are distinct points x and y such that either
> a) R contains precisely those points z such that z Temp-Bet xy and not ($z \approx_{Temp} x$) and not ($z \approx_{Temp} y$); or
> b) R contains precisely those points z such that y Temp-Bet xz and not ($z \approx_{Temp} y$).
> A function is continuous at c if, for any temperature-basic region that contains c, there is a spatio-temporally basic subregion that contains c.

Given these definitions, Field has to make sure that he can provide sufficient regions of space-time to model the replacement axiom.
In order to do that, he assumes space-time regions and a mereological logic which allows us to combine them.
Field's assumption of mereology yields a space-time thick enough to do the work of the real numbers.
Unfortunately, we can also construct physical analogues of statements like the continuum hypothesis and the axiom of choice.
Thus in the second-order reformulation, these statements will be settled on the basis of physical geometry, despite being open questions mathematically.
Settling the size of the continuum would be counterintuitive, given the Gödel-Cohen independence results.

More unsettling, Field's second-order theory is Gödel-incomplete, and so unacceptable to the defender of the indispensability argument.
There is a true statement of the language of the second-order theory, underivable within that theory, a Gödel sentence for the theory.
In contrast, it is a requirement for the indispensability argument that the theory in which we find our commitments must be complete
The equivalence of semantic consequence and derivability entails that there are no underivable truths in a complete theory.
The incompleteness of the second-order theory renders it unfit to do the work that QI requires.
Field tries to minimize the importance of the omissions; the Gödel sentence isn't physically significant.

> I suspect that the extra strength that [the platonist theory] has over [the nominalist theory] is confined to such recherché consequences... (Field: 104).

The class of sentences provable by adding set theory to the nominalist theory is larger than this case.

A version of the Banach-Tarski paradox may be constructed in Field's theory.
The Banach-Tarski paradox is that a region consisting of a solid ball of unit radius can be decomposed into finitely many parts and rearranged to form a solid ball of twice the radius.
As a theorem of pure mathematics, this is counter-intuitive, but not repugnant.
As a theorem about physical space, it is impossible to accept.

Field's reformulations, whether first- or second-order, do not provide the clean representation theorems in a neutral first-order theory that would decisively contradict QI4.
John Burgess, following Field, used something called a two-sorted first-order logic to construct the representation theorems that Field sought.
In a two-sorted theory, there are two different kinds of variables.
In Burgess's reformulation, primary variables range over physical entities and secondary variables range over mathematical entities.
To generate the appropriate representation theorem, Burgess shows how to eliminate all secondary and mixed formulas from the scientific theory.
Burgess provides a mapping from the traditional two-sorted scientific theory to a one-sorted theory which contains only the primary primitives and axioms of the original theory, and counterparts of each of the secondary and mixed primitives and axioms.
This mapping provides the attractive reformulation Field sought for NGT.
Adequate reformulations for some physical theories which need not violate any of the ground rules are thus available.

Despite Burgess's work, questions remain whether such constructions will be available for other physical theories.
NGT is a simple, and outdated, physical theory.
It will be much more difficult to remove mathematics from current and future theories more sophisticated than NGT.

VI. Conservativeness

Recall that Field's project has two parts: the representation theorems and the argument for conservativeness.
Field provides two kinds of arguments for conservativeness, which we can call top-down and bottom-up.

The top-down argument alleges that set theory is conservative over any nominalistically acceptable body of assertions.
The top-down argument is broad.
It applies to all set theory, which Field assumes is a basis for all of mathematics, over any physical theory.
Field's top-down argument consists of showing, both model-theoretically and proof-theoretically, the proximity of conservativeness and consistency.
In other work, Field calls conservativeness a generalized form of consistency.

To establish top-down conservativeness, Field relies on a proof that adding mathematics to a consistent nominalist theory yields a consistent theory.
Field shows how to construct a model for a nominalist theory when it is supplemented with mathematical axioms.

Field's top-down argument shows that adding mathematics to a nominalist theory will not generate a contradiction unless that theory is already inconsistent.
But, Field's top-down argument is insufficient to establish conservativeness.
His proof only shows that the addition of mathematics does not make a consistent theory inconsistent.
Conservativeness entails that adding mathematics to a nominalist theory should not license new conclusions.[1]

Field's bottom-up arguments for conservativeness rely on his representation theorems.
The bottom-up arguments demonstrate the conservativeness of particular mathematical theories over particular scientific theories.
Field's representation theorems map space-time points onto real numbers, showing how to translate nominalist statements into statements about the real numbers.
If Field's representation theorems are available, the conservativeness of the theory of real numbers over NGT follows directly.

While the top-down argument for conservativeness fails, the bottom-up argument, through the representation theorems, works in some specific cases.
But the bottom-up arguments do not establish the conservativeness of mathematics generally, as Field originally claimed.
Since conservativeness is not generally established, Field does not provide a comprehensive response to Quine's indispensability argument.
Still, Field provides a strategy for the nominalist to follow in order to establish conservativeness: construct representation theorems for particular viable physical theories.
The question then becomes whether such theorems are available.
Much of the critical discussion of Field's work centered on this question.

David Malament doubted the likelihood of developing reformulations of classical Hamiltonian mechanics, a phase-space theory, and of quantum mechanics (QM).
The problem with phase-space theories is that the quantifiers normally range over mathematical models, which are not nominalistically acceptable.
A reformulation of QM has to quantify over points and regions of Hilbert space, and propositions, sets of quantum events.
Mark Balaguer, beginning a dispensabilist project for QM, takes these sets of quantum events as physically real propensity properties of quantum systems.
The properties can either be taken as nominalistically acceptable, or one may nominalize them in the way that Field nominalized geometric properties, like length.
Prima facie, a propensity property seems like the kind of abstract object a nominalist would abhor.
Even if Balaguer's technique is successful, there are other physical theories which we do not know how to nominalize, and future physical theories will bring their own difficulties.

For general relativity, for example, there is no available suitable version of the required geometry, as Hilbert's geometry was for Field's project.
Relativity uses a curved space-time, in contrast to the flat space-time underlying NGT.

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[1] Honestly, this seems like such an obvious mistake that I'm sure that I'm missing or misunderstanding something about Field's argument in the appendix to Chapter 1. I made this argument in my dissertation, though, and none of my readers saw a problem with my claim.

Field was originally optimistic about the availability of a synthetic geometry for space-times which are not flat.
But, no one has developed such a geometry yet.

On another front, Michael Resnik argues that the dispensabilist has no account of statistical inference.
Field has not given us a way to account for scientific inferences involving probabilities.
Psychology and economics, for example, are full of problematic inductive inferences.
Physics may be easier to nominalize, just because it is so mathematized.
The close connection between physical geometry and pure geometry make a nominalist formulation more likely.

Pronouncements on the likelihood of extending Field's strategy are doubly speculative.
We do not know the nature of our ideal physical theory.
And we do not know what kinds of dispensabilist techniques may be developed.
Burgess argues that such limitations are no reason to be skeptical of their eventual availability.

> As a consequence of nominalism's being mainly a philosopher's concern, this open research problem is...one that has so far been investigated only by amateurs - philosophers and logicians - not professionals - geometers and physicists; and the failure of amateurs to surmount the obstacles is no strong grounds for pessimism about what could be achieved by professionals (Burgess and Rosen 1997: 118).

Worries about whether dispensabilist constructions like Field's are or will be available for future science are thus indecisive.

VII. Logic Matters

Field's 1980 monograph took over the philosophy of mathematics for over a decade.
Even now, the success of Field's project is seen by many philosophers as the determining question in the philosophy of mathematics.
Much of Field's subsequent discussion about his project centered around two questions.
First, he defended substantivalism about space-time points.
Second, he defended his views of metalogic.
Since Field's representation theorems ran into difficulties over their logic, and since some other questions about mathematics arise from considering its inferential role, it will be instructive to look a little at Field's discussion of logic, and its role in accounting for what we ordinarily take to be mathematical knowledge.
To support his claim that we can eliminate mathematical objects from our ontology, Field not only provided a reformulation of standard science, but also an account of what we ordinarily take to be our mathematical knowledge, including knowledge of mathematical inferences.

Field denies both that science needs numbers and that we have any mathematical knowledge.
But we seem to have lots of mathematical knowledge.
We seem to know mathematical axioms, for example, and that certain theorems follow from them.
We also seem to know of the consistency of sets of (especially mathematical) sentences.
To support his fictionalism, Field accounts for what is ordinarily seen as mathematical knowledge in three ways.

First, he argues that much of what we take to be mathematical knowledge is really empirical knowledge, of space-time for example.
In addition, some empirical knowledge about which axioms are generally accepted by the mathematical community, and which problems are seen as important, might be taken for mathematical knowledge.

Second, though only a surprisingly small part of the account, Field claims that, for limited results, there is no knowledge to be had.
Some debatable axioms, like the axiom of measurable cardinals, are not derivative of other commonly held axioms, so they can not be known on the basis of inference.
Neither can they be taken to be known empirically, since they correspond to no physical facts.

Lastly and mostly, Field argues that much of what we take to be mathematical knowledge is really logical knowledge of which theorems follow from which axioms.
Field thus defends some degree of mathematical objectivity, without committing to mathematical objects, on the basis of logical objectivity.

> [Y]ou don't need to make mathematics actually be about anything for it to be possible to objectively assess the logical relations between mathematical premises and mathematical conclusions (Field 1998a: 317).

Knowledge of inferences is ordinarily taken to be meta-theoretic knowledge
Thus, Field paid special attention to metalogic, the study of logical inferences and entailments.
Metalogic is a challenge for the nominalist, since standard formulations invoke lots of set theory.
One option is to deny that our uses of set theory commit us to the existence of mathematical objects.
But, this instrumentalist interpretation is not consistent with Field's views.
Remember, Field is adopting the method underlying Quine's indispensability argument, including his double-talk criticism of Carnap.

> If one *just* advocates fictionalism about a portion of mathematics, without showing how that part of mathematics is dispensable in applications, then one is engaging in intellectual doublethink: one is merely taking back in one's philosophical moments what one asserts in doing science... (Field 2).

The whole point of Field's reformulation of NGT was to avoid such doublethink,

Another way to account for metalogic would use a construction like the one that he provides for physics.
In other words, he could try to eliminate set theory from metalogic by reformulating metalogical theories to quantify over space-time points too.
It is easy to see why Field would avoid such an empirical account.
If metalogical claims were known empirically, then they would be factual.
Given his close connection between mathematical knowledge and logical knowledge, this could lead to truth values for mathematical existence claims.

Instead, Field uses a limited version of modal logic to account for our knowledge of consistency.
Formally, Field introduces an object-level modal operator to represent consistency, treating our knowledge of consistency like that of conjunction and negation.
Defending this modal logic, and his views about logic more generally, has occupied much of the last thirty years of his work.

Field's replacement of mathematical knowledge with logical knowledge has received less attention than his replacement of mathematical objects with space-time points.
Yet even without a successful reformulation of mathematics, Field may have accounted for much of what we take to be mathematical knowledge without commitment to mathematical objects.

VIII. Ideology

Some philosophers do not see Field's dispensabilist reformulation as eliminating mathematics, even if it abides by the Ground Rules.
Such critics maintain that Field's theory includes mathematics, despite lacking quantification over mathematical objects.
The idea behind this criticism is that eliminating quantification over mathematical objects in physical theories does not suffice to eliminate mathematics from our ontology.

There are two ways in which mathematics might be said to remain in Field's nominalist theory.
First, we might claim that space-time points are really mathematical objects.
Michael Resnik made this criticism.

> Owing to the richness of Field's physical ontology, philosophers in Quine's tradition might object that Field has just hidden his mathematical objects in physical disguises (Resnik 1985b: 192).

So did Penelope Maddy.

> [In space-time, we have] everything essential to the real numbers (Maddy 1990b: 201).

We could further erode the differences between real numbers and space-time points by arguing that mathematical objects are really empirical posits, as Quine does.
Field insists that his construction makes a difference to ontology.

> It is hardly surprising that mathematical theories developed in order to apply to space and time should postulate mathematical structures with some strong structural similarities to the physical structures of space and time. It is a clear case of putting the cart before the horse to conclude from this that what I've called the physical structure of space and time is really mathematical structure in disguise (Field 33-34).

Second, we might claim that the ideology that goes with both mathematics and Field's nominalist theory is what really matters to mathematics.
Even ridding physical science of mathematical ontology, says the defend of the second argument, does not rid science of mathematical ideology.
Ontology, in this sense, is what the variables express.
Ideology is embodied in the predicates.
Both Field's space-time ontology and the standard mathematical ontology contain continuous objects, for example.
Both kinds of objects satisfy the mathematical properties required by science.

Field argues that while the reformulation does have an ideology that is similar to mathematical ideology, it is also importantly different.

> Postulating physical space isn't like postulating real numbers...the ideology that goes with the postulate of points of space is less rich than that which goes with the postulate of the real numbers (Field 32).

The richness of mathematics includes the un-applied portions which Quine called recreation.
For Field, we need not construct representation theorems for such un-applied portions of mathematics.

IX. Assessing the Project

We have not looked at Field's project in great detail.
It is a remarkable technical project.
Some of the work it inspired is even more impressive.
The question is whether it shows that the indispensability argument fails to justify our beliefs in mathematical objects.
I believe that it does not.
I also believe that the indispensability argument never justified our beliefs in mathematical objects in the first place.
Field's reformulation, and even considering Burgess's improvements, violates his own Ground Rules.

Field claims to have produced an adequate, logically neutral, attractive reformulation on which to base an argument for the conservativeness of mathematics.
But he uses an incomplete logic, and his logical extensions undermine his claim to semantic uniformity.

> One reason for [its failure] is that Field is forced to make such strong extensions of standard first-order logic that it is no longer possible to assess the nominalist nature of his theory solely by the range of its bound variables (Resnik 1985a: 163).

Field urges that we not take mathematical propositions as true claims containing oblique references to concrete objects because doing so makes explaining the applicability of mathematics to science more difficult.
If we adjust the logic, we are no longer taking the sentences at face value.
I conclude:

> FC.1: Field's project extends logic unacceptably, and there are no similar constructions for current and future scientific theories.
> FC.2: Field misapplies an attractiveness criterion for the acceptability of dispensabilist reformulations.
> FC.3: Conservativeness, as Field originally imagines it, does not hold generally, though a restricted version does hold.

The nominalist looks at Field's project and sees hope for removing mathematics from scientific theory.
The platonist sees the failures to remove mathematics from the scientific theories we deem most important.